Matrix Norms

Tom Lyche

Centre of Mathematics for Applications,
Department of Informatics,
University of Oslo

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Matrix Norms

We consider matrix norms on $(\mathbb{C}^{m,n}, \mathbb{C})$. All results holds for $(\mathbb{R}^{m,n}, \mathbb{R})$.

Definition (Matrix Norms)

A function $\|\cdot\|: \mathbb{C}^{m,n} \to \mathbb{C}$ is called a **matrix norm** on $\mathbb{C}^{m,n}$ if for all $A, B \in \mathbb{C}^{m,n}$ and all $\alpha \in \mathbb{C}$

1. $\|A\| \geq 0$ with equality if and only if $A = 0$. (positivity)
2. $\|\alpha A\| = |\alpha| \, \|A\|$. (homogeneity)
3. $\|A + B\| \leq \|A\| + \|B\|$. (subadditivity)

A matrix norm is simply a vector norm on the finite dimensional vector spaces $(\mathbb{C}^{m,n}, \mathbb{C})$ of $m \times n$ matrices.

Equivalent norms

Adapting some general results on vector norms to matrix norms give

Theorem

1. *All matrix norms are equivalent. Thus, if* $\|\cdot\|$ *and* $\|\cdot\|'$ *are two matrix norms on* $\mathbb{C}^{m,n}$ *then there are positive constants* μ *and* M *such that* $\mu\|A\| \leq \|A\|' \leq M\|A\|$ *holds for all* $A \in \mathbb{C}^{m,n}$.
2. *A matrix norm is a continuous function* $\|\cdot\| : \mathbb{C}^{m,n} \to \mathbb{R}$.

Examples

- From any vector norm $\|\ \|_V$ on $\mathbb{C}^{mn}$ we can define a matrix norm on $\mathbb{C}^{m,n}$ by $\|\mathbf{A}\| := \|\text{vec}(\mathbf{A})\|_V$, where $\text{vec}(\mathbf{A}) \in \mathbb{C}^{mn}$ is the vector obtained by stacking the columns of $\mathbf{A}$ on top of each other.
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$$\begin{aligned}
\|\mathbf{A}\|_S &:= \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|,\ \ p=1,\ \textbf{Sum norm},\\
\|\mathbf{A}\|_F &:= \Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{1/2},\ \ p=2,\ \textbf{Frobenius norm},\\
\|\mathbf{A}\|_M &:= \max_{i,j}|a_{ij}|,\ \ p=\infty,\ \textbf{Max norm}.
\end{aligned} \tag{1}$$

The Frobenius Matrix Norm 1.

- $\|\mathbf{A}^H\|_F^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} |\overline{a}_{ij}|^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = \|\mathbf{A}\|_F^2$.

The Frobenius Matrix Norm 2.

- $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{m} \|\mathbf{a}_{i\cdot}\|_2^2$
- $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}|^2 = \sum_{j=1}^{n} \|\mathbf{a}_{\cdot j}\|_2^2$.

Unitary Invariance.

- If $\mathbf{A} \in \mathbb{C}^{m,n}$ and $\mathbf{U} \in \mathbb{C}^{m,m}$, $\mathbf{V} \in \mathbb{C}^{n,n}$ are unitary
- $\|\mathbf{UA}\|_F^2 \overset{2.}{=} \sum_{j=1}^{n} \|\mathbf{Ua}_{\cdot j}\|_2^2 = \sum_{j=1}^{n} \|\mathbf{a}_{\cdot j}\|_2^2 \overset{2.}{=} \|\mathbf{A}\|_F^2$.
- $\|\mathbf{AV}\|_F \overset{1.}{=} \|\mathbf{V}^H\mathbf{A}^H\|_F = \|\mathbf{A}^H\|_F \overset{1.}{=} \|\mathbf{A}\|_F$.

Submultiplicativity

- Suppose $\mathbf{A}, \mathbf{B}$ are rectangular matrices so that the product $\mathbf{AB}$ is defined.
- $\|\mathbf{AB}\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{k} \left(\mathbf{a}_{i\cdot}^T \mathbf{b}_{\cdot j}\right)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{k} \|\mathbf{a}_{i\cdot}\|_2^2 \|\mathbf{b}_{\cdot j}\|_2^2 = \|\mathbf{A}\|_F^2 \|\mathbf{B}\|_F^2.$

Subordinance

- $\|\mathbf{Ax}\|_2 \leq \|\mathbf{A}\|_F \|\mathbf{x}\|_2$, for all $\mathbf{x} \in \mathbb{C}^n$.
- Since $\|\mathbf{v}\|_F = \|\mathbf{v}\|_2$ for a vector this follows from submultiplicativity.

Explicit Expression

- Let $\mathbf{A} \in \mathbb{C}^{m,n}$ have singular values $\sigma_1, \ldots, \sigma_n$ and SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$. Then
- $\|\mathbf{A}\|_F \overset{3.}{=} \|\mathbf{U}^H\mathbf{A}\mathbf{V}\|_F = \|\mathbf{\Sigma}\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$.

Consistency

- A matrix norm is called **consistent on** $\mathbb{C}^{n,n}$ if
  $$4. \quad \|\mathbf{AB}\| \le \|\mathbf{A}\| \, \|\mathbf{B}\| \qquad \text{(submultiplicativity)}$$
  holds for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n,n}$.
- A matrix norm is **consistent** if it is defined on $\mathbb{C}^{m,n}$ for all $m, n \in \mathbb{N}$, and 4. holds for all matrices $\mathbf{A}, \mathbf{B}$ for which the product $\mathbf{AB}$ is defined.
- The Frobenius norm is consistent.
- The Sum norm is consistent.
- The Max norm is not consistent.
- The norm $\|\mathbf{A}\| := \sqrt{mn}\|\mathbf{A}\|_M, \quad \mathbf{A} \in \mathbb{C}^{m,n}$ is consistent.

Subordinate Matrix Norm

Definition

- Suppose $m, n \in \mathbb{N}$ are given,
- Let $\| \ \|_\alpha$ on $\mathbb{C}^m$ and $\| \ \|_\beta$ on $\mathbb{C}^n$ be vector norms, and let $\| \ \|$ be a matrix norm on $\mathbb{C}^{m,n}$.
- We say that the matrix norm $\| \ \|$ is **subordinate** to the vector norms $\| \ \|_\alpha$ and $\| \ \|_\beta$ if $\|\mathbf{A}\mathbf{x}\|_\alpha \leq \|\mathbf{A}\| \, \|\mathbf{x}\|_\beta$ for all $\mathbf{A} \in \mathbb{C}^{m,n}$ and all $\mathbf{x} \in \mathbb{C}^n$.
- If $\| \ \|_\alpha = \| \ \|_\beta$ then we say that $\| \ \|$ is subordinate to $\| \ \|_\alpha$.

- The Frobenius norm is subordinate to the Euclidian vector norm.
- The Sum norm is subordinate to the l_1-norm.
- $\|\mathbf{A}\mathbf{x}\|_\infty \leq \|\mathbf{A}\|_M \|\mathbf{x}\|_1$.

Operator Norm

Definition

Suppose $m, n \in \mathbb{N}$ are given and let $\|\cdot\|_\alpha$ be a vector norm on $\mathbb{C}^n$ and $\|\cdot\|_\beta$ a vector norm on $\mathbb{C}^m$. For $A \in \mathbb{C}^{m,n}$ we define

$$\|A\| := \|A\|_{\alpha,\beta} := \max_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_\beta}{\|\mathbf{x}\|_\alpha}. \tag{2}$$

We call this the (α, β) **operator norm**, the (α, β)-norm, or simply the α-norm if $\alpha = \beta$.

Observations

- $\|\mathbf{A}\|_{\alpha,\beta} = \max_{\mathbf{x}\notin \ker(\mathbf{A})} \frac{\|\mathbf{Ax}\|_\alpha}{\|\mathbf{x}\|_\beta} = \max_{\|\mathbf{x}\|_\beta=1} \|\mathbf{Ax}\|_\alpha$.
- $\|\mathbf{Ax}\|_\alpha \leq \|\mathbf{A}\|\|\mathbf{x}\|_\beta$.
- $\|\mathbf{A}\|_{\alpha,\beta} = \|\mathbf{Ax}^*\|_\alpha$ for some $\mathbf{x}^* \in \mathbb{C}^n$ with $\|\mathbf{x}^*\|_\beta = 1$.
- The operator norm is a matrix norm on $\mathbb{C}^{mn,}$.
- The Sum norm and Frobenius norm are not an α operator norm for any α.

Operator norm Properties

- The operator norm is a matrix norm on $\mathbb{C}^{m,n}$.
- The operator norm is consistent if the vector norm $\|\ \|_\alpha$ is defined for all $m \in \mathbb{N}$ and $\|\ \|_\beta = \|\ \|_\alpha$.

Proof

In 2. and 3. below we take the max over the unit sphere $\mathcal{S}_\beta$.

1. Nonnegativity is obvious. If $\|\mathbf{A}\| = 0$ then $\|\mathbf{Ay}\|_\beta = 0$ for each $\mathbf{y} \in \mathbb{C}^n$. In particular, each column $\mathbf{Ae}_j$ in $\mathbf{A}$ is zero. Hence $\mathbf{A} = 0$.
2. $\|c\mathbf{A}\| = \max_{\mathbf{x}} \|c\mathbf{Ax}\|_\alpha = \max_{\mathbf{x}} |c| \, \|\mathbf{Ax}\|_\alpha = |c| \, \|\mathbf{A}\|$.
3. $\|\mathbf{A} + \mathbf{B}\| = \max_{\mathbf{x}} \|(\mathbf{A} + \mathbf{B})\mathbf{x}\|_\alpha \leq$ $\max_{\mathbf{x}} \|\mathbf{Ax}\|_\alpha + \max_{\mathbf{x}} \|\mathbf{Bx}\|_\alpha = \|\mathbf{A}\| + \|\mathbf{B}\|$.
4. $\|\mathbf{AB}\| = \max_{\mathbf{Bx} \neq \mathbf{0}} \frac{\|\mathbf{ABx}\|_\alpha}{\|\mathbf{x}\|_\alpha} = \max_{\mathbf{Bx} \neq \mathbf{0}} \frac{\|\mathbf{ABx}\|_\alpha}{\|\mathbf{Bx}\|_\alpha} \frac{\|\mathbf{Bx}\|_\alpha}{\|\mathbf{x}\|_\alpha}$
$$\leq \max_{\mathbf{y} \neq \mathbf{0}} \frac{\|\mathbf{Ay}\|_\alpha}{\|\mathbf{y}\|_\alpha} \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Bx}\|_\alpha}{\|\mathbf{x}\|_\alpha} = \|\mathbf{A}\| \, \|\mathbf{B}\|.$$

The p matrix norm

- The operator norms $\|\cdot\|_p$ defined from the p-vector norms are of special interest.
- Recall
  $\|\mathbf{x}\|_p := \left(\sum_{j=1}^n |x_j|^p\right)^{1/p},\ p \geq 1, \quad \|\mathbf{x}\|_\infty := \max_{1\leq j\leq n}|x_j|.$
- Used quite frequently for $p = 1, 2, \infty$.
- We define for any $1 \leq p \leq \infty$

$$\|A\|_p := \max_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{y}\|_p=1} \|A\mathbf{y}\|_p. \tag{3}$$

- The p-norms are consistent matrix norms which are subordinate to the p-vector norm.

Explicit expressions

Theorem

For $\mathbf{A} \in \mathbb{C}^{m,n}$ *we have*

$$\begin{aligned}
\|\mathbf{A}\|_1 &:= \max_{1\le j\le n} \sum_{k=1}^{m} |a_{k,j}|, && \textit{(max column sum)}\\
\|\mathbf{A}\|_2 &:= \sigma_1, && \textit{(largest singular value of } \mathbf{A}\textit{)}\\
\|\mathbf{A}\|_\infty &:= \max_{1\le k\le m} \sum_{j=1}^{n} |a_{k,j}|, && \textit{(max row sum).}
\end{aligned} \tag{4}$$

The expression $\|\mathbf{A}\|_2$ is called the **two-norm** or the **spectral norm** of **A**. The explicit expression follows from the minmax theorem for singular values.

Examples

For $\mathbf{A} := \frac{1}{15}\begin{bmatrix} 14 & 4 & 16 \\ 2 & 22 & 13 \end{bmatrix}$ we find

- $\|\mathbf{A}\|_1 = \frac{29}{15}$.
- $\|\mathbf{A}\|_2 = 2$.
- $\|\mathbf{A}\|_\infty = \frac{37}{15}$.
- $\|\mathbf{A}\|_F = \sqrt{5}$.
- $A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- $\|A\|_1 = 6$
- $\|A\|_2 = 5.465$
- $\|A\|_\infty = 7.$
- $\|A\|_F = 5.4772$

The 2 norm

Theorem

Suppose $\mathbf{A} \in \mathbb{C}^{n,n}$ *has singular values* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ *and eigenvalues* $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. *Then*

$$\|\mathbf{A}\|_2 = \sigma_1 \text{ and } \|\mathbf{A}^{-1}\|_2 = \frac{1}{\sigma_n}, \tag{5}$$

$$\|\mathbf{A}\|_2 = \lambda_1 \text{ and } \|\mathbf{A}^{-1}\|_2 = \frac{1}{\lambda_n}, \quad \text{if } \mathbf{A} \text{ is symmetric positive definite,} \tag{6}$$

$$\|\mathbf{A}\|_2 = |\lambda_1| \text{ and } \|\mathbf{A}^{-1}\|_2 = \frac{1}{|\lambda_n|}, \text{ if } \mathbf{A} \text{ is normal.} \tag{7}$$

For the norms of $\mathbf{A}^{-1}$ *we assume of course that* $\mathbf{A}$ *is nonsingular.*

Unitary Transformations

Definition

A matrix norm $\| \ \|$ on $\mathbb{C}^{m,n}$ is called **unitary invariant** if $\|\mathbf{UAV}\| = \|\mathbf{A}\|$ for any $\mathbf{A} \in \mathbb{C}^{m,n}$ and any unitary matrices $\mathbf{U} \in \mathbb{C}^{m,m}$ and $\mathbf{V} \in \mathbb{C}^{n,n}$.

If $\mathbf{U}$ and $\mathbf{V}$ are unitary then $\mathbf{U}(\mathbf{A}+\mathbf{E})\mathbf{V} = \mathbf{UAV} + \mathbf{F}$, where $\|\mathbf{F}\| = \|\mathbf{E}\|$.

Theorem

The Frobenius norm and the spectral norm are unitary invariant. Moreover $\|\mathbf{A}^H\|_F = \|\mathbf{A}\|_F$ *and* $\|\mathbf{A}^H\|_2 = \|\mathbf{A}\|_2$.

Proof 2 norm

- $\|\mathbf{UA}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{UAx}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{Ax}\|_2 = \|\mathbf{A}\|_2$.
- $\|\mathbf{A}^H\|_2 = \|\mathbf{A}\|_2$ (same singular values).
- $\|\mathbf{AV}\|_2 = \|(\mathbf{AV})^H\|_2 = \|\mathbf{V}^H\mathbf{A}^H\|_2 = \|\mathbf{A}^H\|_2 = \|\mathbf{A}\|_2$.

Perturbation of linear systems

- $$\begin{array}{rcrcl} x_1 & + & x_2 & = & 20 \\ x_1 & + & (1-10^{-16})x_2 & = & 20-10^{-15} \end{array}$$
- The exact solution is $x_1 = x_2 = 10$.
- Suppose we replace the second equation by

$$x_1 + (1+10^{-16})x_2 = 20 - 10^{-15},$$

- the exact solution changes to $x_1 = 30$, $x_2 = -10$.
- A small change in one of the coefficients, from $1 - 10^{-16}$ to $1 + 10^{-16}$, changed the exact solution by a large amount.

Ill Conditioning

- A mathematical problem in which the solution is very sensitive to changes in the data is called **ill-conditioned** or sometimes **ill-posed**.
- Such problems are difficult to solve on a computer.
- If at all possible, the mathematical model should be changed to obtain a more well-conditioned or properly-posed problem.

Perturbations

- We consider what effect a small change (perturbation) in the data A,$\mathbf{b}$ has on the solution $\mathbf{x}$ of a linear system $A\mathbf{x} = \mathbf{b}$.
- Suppose $\mathbf{y}$ solves $(A + E)\mathbf{y} = \mathbf{b}+\mathbf{e}$ where E is a (small) $n \times n$ matrix and $\mathbf{e}$ a (small) vector.
- How large can $\mathbf{y}-\mathbf{x}$ be?
- To measure this we use vector and matrix norms.

Conditions on the norms

- $\|\cdot\|$ will denote a vector norm on $\mathbb{C}^n$ and also a submultiplicative matrix norm on $\mathbb{C}^{n,n}$ which in addition is subordinate to the vector norm.
- Thus for any $A, B \in \mathbb{C}^{n,n}$ and any $\mathbf{x} \in \mathbb{C}^n$ we have

$$\|AB\| \leq \|A\| \, \|B\| \text{ and } \|A\mathbf{x}\| \leq \|A\| \, \|\mathbf{x}\|.$$

- This is satisfied if the matrix norm is the operator norm corresponding to the given vector norm or the Frobenius norm.

Absolute and relative error

- The difference $\|\mathbf{y} - \mathbf{x}\|$ measures the absolute error in $\mathbf{y}$ as an approximation to $\mathbf{x}$,
- $\|\mathbf{y} - \mathbf{x}\| / \|\mathbf{x}\|$ or $\|\mathbf{y} - \mathbf{x}\| / \|\mathbf{y}\|$ is a measure for the relative error.

Perturbation in the right hand side

Theorem

Suppose $A \in \mathbb{C}^{n,n}$ *is invertible,* $\mathbf{b}, \mathbf{e} \in \mathbb{C}^n$, $\mathbf{b} \neq \mathbf{0}$ *and* $A\mathbf{x} = \mathbf{b}$, $A\mathbf{y} = \mathbf{b} + \mathbf{e}$. *Then*

$$\frac{1}{K(A)} \frac{\|\mathbf{e}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{y} - \mathbf{x}\|}{\|\mathbf{x}\|} \leq K(A) \frac{\|\mathbf{e}\|}{\|\mathbf{b}\|}, \quad K(A) = \|A\| \|A^{-1}\|. \tag{8}$$

- Proof:
- Consider (8). $\|\mathbf{e}\|/\|\mathbf{b}\|$ is a measure for the size of the perturbation $\mathbf{e}$ relative to the size of $\mathbf{b}$. $\|\mathbf{y} - \mathbf{x}\|/\|\mathbf{x}\|$ can in the worst case be

$$K(A) = \|A\| \|A^{-1}\|$$

times as large as $\|\mathbf{e}\|/\|\mathbf{b}\|$.

Condition number

- $K(A)$ is called the **condition number with respect to inversion of a matrix**, or just the condition number, if it is clear from the context that we are talking about solving linear systems.
- The condition number depends on the matrix A and on the norm used. If $K(A)$ is large, A is called **ill-conditioned** (with respect to inversion).
- If $K(A)$ is small, A is called **well-conditioned** (with respect to inversion).

Condition number properties

- Since $\|A\|\|A^{-1}\| \geq \|AA^{-1}\| = \|I\| \geq 1$ we always have $K(A) \geq 1$.
- Since all matrix norms are equivalent, the dependence of $K(A)$ on the norm chosen is less important than the dependence on A.
- Usually one chooses the spectral norm when discussing properties of the condition number, and the l_1 and l_∞ norm when one wishes to compute it or estimate it.

The 2-norm

- Suppose $\mathbf{A}$ has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ if $\mathbf{A}$ is square.
- $K_2(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$
- $K_2(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 = \frac{|\lambda_1|}{|\lambda_n|}, \quad \mathbf{A} \text{ normal.}$
- It follows that $\mathbf{A}$ is ill-conditioned with respect to inversion if and only if σ_1/σ_n is large, or $|\lambda_1|/|\lambda_n|$ is large when $\mathbf{A}$ is normal.
- $K_2(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 = \frac{\lambda_1}{\lambda_n}, \quad \mathbf{A} \text{ positive definite.}$

The residual

Suppose we have computed an approximate solution $\mathbf{y}$ to $\mathbf{Ax} = \mathbf{b}$. The vector $\mathbf{r}(\mathbf{y} :) = \mathbf{Ay} - \mathbf{b}$ is called the residual vector , or just the residual. We can bound $\mathbf{x} - \mathbf{y}$ in term of $\mathbf{r}(\mathbf{y})$.

Theorem

Suppose $\mathbf{A} \in \mathbb{C}^{n,n}$, $\mathbf{b} \in \mathbb{C}^n$, $\mathbf{A}$ is nonsingular and $\mathbf{b} \neq \mathbf{0}$. Let $\mathbf{r}(\mathbf{y}) = \mathbf{Ay} - \mathbf{b}$ for each $\mathbf{y} \in \mathbb{C}^n$. If $\mathbf{Ax} = \mathbf{b}$ then

$$\frac{1}{K(\mathbf{A})} \frac{\|\mathbf{r}(\mathbf{y})\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{y} - \mathbf{x}\|}{\|\mathbf{x}\|} \leq K(\mathbf{A}) \frac{\|\mathbf{r}(\mathbf{y})\|}{\|\mathbf{b}\|}. \tag{9}$$

Discussion

- If $\mathbf{A}$ is well-conditioned, (9) says that $\|\mathbf{y} - \mathbf{x}\| / \|\mathbf{x}\| \approx \|\mathbf{r}(\mathbf{y})\| / \|\mathbf{b}\|$.
- In other words, the accuracy in $\mathbf{y}$ is about the same order of magnitude as the residual as long as $\|\mathbf{b}\| \approx 1$.
- If $\mathbf{A}$ is ill-conditioned, anything can happen.
- The solution can be inaccurate even if the residual is small
- We can have an accurate solution even if the residual is large.

The inverse of $\mathbf{A} + \mathbf{E}$

Theorem

Suppose $\mathbf{A} \in \mathbb{C}^{n,n}$ *is nonsingular and let* $\|\cdot\|$ *be a consistent matrix norm on* $\mathbb{C}^{n,n}$. *If* $\mathbf{E} \in \mathbb{C}^{n,n}$ *is so small that* $r := \|\mathbf{A}^{-1}\mathbf{E}\| < 1$ *then* $\mathbf{A} + \mathbf{E}$ *is nonsingular and*

$$\|(\mathbf{A} + \mathbf{E})^{-1}\| \leq \frac{\|\mathbf{A}^{-1}\|}{1 - r}. \tag{10}$$

If $r < 1/2$ *then*

$$\frac{\|(\mathbf{A} + \mathbf{E})^{-1} - \mathbf{A}^{-1}\|}{\|\mathbf{A}^{-1}\|} \leq 2K(\mathbf{A})\frac{\|\mathbf{E}\|}{\|\mathbf{A}\|}. \tag{11}$$

Proof

- We use that if $\mathbf{B} \in \mathbb{C}^{n,n}$ and $\|\mathbf{B}\| < 1$ then $\mathbf{I} - \mathbf{B}$ is nonsingular and $\|(\mathbf{I} - \mathbf{B})^{-1}\| \leq \frac{1}{1-\|\mathbf{B}\|}$.
- Since $r < 1$ the matrix $\mathbf{I} - \mathbf{B} := \mathbf{I} + \mathbf{A}^{-1}\mathbf{E}$ is nonsingular.
- Since $(\mathbf{I} - \mathbf{B})^{-1}\mathbf{A}^{-1}(\mathbf{A} + \mathbf{E}) = \mathbf{I}$ we see that $\mathbf{A} + \mathbf{E}$ is nonsingular with inverse $(\mathbf{I} - \mathbf{B})^{-1}\mathbf{A}^{-1}$.
- Hence, $\|(\mathbf{A} + \mathbf{E})^{-1}\| \leq \|(\mathbf{I} - \mathbf{B})^{-1}\|\|\mathbf{A}^{-1}\|$ and (10) follows.
- From the identity $(\mathbf{A} + \mathbf{E})^{-1} - \mathbf{A}^{-1} = -\mathbf{A}^{-1}\mathbf{E}(\mathbf{A} + \mathbf{E})^{-1}$ we obtain by $\|(\mathbf{A} + \mathbf{E})^{-1} - \mathbf{A}^{-1}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{E}\|\|(\mathbf{A} + \mathbf{E})^{-1}\| \leq K(\mathbf{A})\frac{\|\mathbf{E}\|}{\|\mathbf{A}\|}\frac{\|\mathbf{A}^{-1}\|}{1-r}$.
- Dividing by $\|\mathbf{A}^{-1}\|$ and setting $r = 1/2$ proves (11).

Perturbation in $\mathbf{A}$

Theorem

Suppose $\mathbf{A}, \mathbf{E} \in \mathbb{C}^{n,n}$, $\mathbf{b} \in \mathbb{C}^n$ *with* $\mathbf{A}$ *invertible and* $\mathbf{b} \neq \mathbf{0}$. *If* $r := \|\mathbf{A}^{-1}\mathbf{E}\| < 1/2$ *for some operator norm then* $\mathbf{A} + \mathbf{E}$ *is invertible. If* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *and* $(\mathbf{A} + \mathbf{E})\mathbf{y} = \mathbf{b}$ *then*

$$\frac{\|\mathbf{y} - \mathbf{x}\|}{\|\mathbf{y}\|} \leq \|\mathbf{A}^{-1}\mathbf{E}\| \leq K(\mathbf{A})\frac{\|\mathbf{E}\|}{\|\mathbf{A}\|}, \tag{12}$$

$$\frac{\|\mathbf{y} - \mathbf{x}\|}{\|\mathbf{x}\|} \leq 2K(\mathbf{A})\frac{\|\mathbf{E}\|}{\|\mathbf{A}\|}.. \tag{13}$$

Proof

- $\mathbf{A} + \mathbf{E}$ is invertible.
- (12) follows easily by taking norms in the equation $\mathbf{x} - \mathbf{y} = \mathbf{A}^{-1}\mathbf{E}\mathbf{y}$ and dividing by $\|\mathbf{y}\|$.
- From the identity $\mathbf{y} - \mathbf{x} = \left((\mathbf{A} + \mathbf{E})^{-1} - \mathbf{A}^{-1}\right)\mathbf{A}\mathbf{x}$ we obtain $\|\mathbf{y} - \mathbf{x}\| \leq \|(\mathbf{A} + \mathbf{E})^{-1} - \mathbf{A}^{-1}\|\|\mathbf{A}\|\|\mathbf{x}\|$ and (13) follows.

Finding the rank of a matrix

- Gauss-Jordan cannot be used to determine rank numerically
- Use singular value decomposition
- numerically will normally find $\sigma_n > 0$.
- Determine minimal r so that $\sigma_{r+1}, \ldots, \sigma_n$ are "close" to round off unit.
- Use this r as an estimate for the rank.

Convergence in $\mathbb{R}^{m,n}$ and $\mathbb{C}^{m,n}$

- Consider an infinite sequence of matrices $\{\mathbf{A}_k\} = \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots$ in $\mathbb{C}^{m,n}$.
- $\{\mathbf{A}_k\}$ is said to converge to the limit $\mathbf{A}$ in $\mathbb{C}^{m,n}$ if each element sequence $\{\mathbf{A}_k(ij)\}_k$ converges to the corresponding element $\mathbf{A}(ij)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
- $\{\mathbf{A}_k\}$ is a **Cauchy sequence** if for each $\epsilon > 0$ there is an integer $N \in \mathbb{N}$ such that for each $k, l \geq N$ and all i, j we have $|\mathbf{A}_k(ij) - \mathbf{A}_l(ij)| \leq \epsilon$.
- $\{\mathbf{A}_k\}$ is bounded if there is a constant M such that $|\mathbf{A}_k(ij)| \leq M$ for all i, j, k.

More on Convergence

- By stacking the columns of $\mathbf{A}$ into a vector in $\mathbb{C}^{mn}$ we obtain
- A sequence $\{\mathbf{A}_k\}$ in $\mathbb{C}^{m,n}$ converges to a matrix $\mathbf{A} \in \mathbb{C}^{m,n}$ if and only if $\lim_{k\to\infty}\|\mathbf{A}_k - \mathbf{A}\| = 0$ for any matrix norm $\|\cdot\|$.
- A sequence $\{\mathbf{A}_k\}$ in $\mathbb{C}^{m,n}$ is convergent if and only if it is a Cauchy sequence.
- Every bounded sequence $\{\mathbf{A}_k\}$ in $\mathbb{C}^{m,n}$ has a convergent subsequence.

The Spectral Radius

- $\rho(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|$.
- For any matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ and any $\mathbf{A} \in \mathbb{C}^{n,n}$ we have $\rho(\mathbf{A}) \leq \|\mathbf{A}\|$.
- Proof: Let $(\lambda, \mathbf{x})$ be an eigenpair for $\mathbf{A}$
- $\mathbf{X} := [\mathbf{x}, \ldots, \mathbf{x}] \in \mathbb{C}^{n,n}$.
- $\lambda \mathbf{X} = \mathbf{A}\mathbf{X}$, which implies $|\lambda| \, \|\mathbf{X}\| = \|\lambda \mathbf{X}\| = \|\mathbf{A}\mathbf{X}\| \leq \|\mathbf{A}\| \, \|\mathbf{X}\|$.
- Since $\|\mathbf{X}\| \neq 0$ we obtain $|\lambda| \leq \|\mathbf{A}\|$.

A Special Norm

Theorem

Let $\mathbf{A} \in \mathbb{C}^{n,n}$ *and* $\epsilon > 0$ *be given. There is a consistent matrix norm* $\|\cdot\|'$ *on* $\mathbb{C}^{n,n}$ *such that* $\rho(\mathbf{A}) \leq \|\mathbf{A}\|' \leq \rho(\mathbf{A}) + \epsilon$.

A Very Important Result

Theorem

For any $\mathbf{A} \in \mathbb{C}^{n,n}$ *we have*

$$\lim_{k\to\infty} \mathbf{A}^k = 0 \iff \rho(\mathbf{A}) < 1.$$

- Proof:
- Suppose $\rho(\mathbf{A}) < 1$.
- There is a consistent matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ such that $\|\mathbf{A}\| < 1$.
- But then $\|\mathbf{A}^k\| \le \|\mathbf{A}\|^k \to 0$ as $k \to \infty$.
- Hence $\mathbf{A}^k \to 0$.
- The converse is easier.

Convergence can be slow

- $\mathbf{A} = \begin{bmatrix} 0.99 & 1 & 0 \\ 0 & 0.99 & 1 \\ 0 & 0 & 0.99 \end{bmatrix}$, $\mathbf{A}^{100} = \begin{bmatrix} 0.4 & 9.37 & 1849 \\ 0 & 0.4 & 37 \\ 0 & 0 & 0.4 \end{bmatrix}$, $\mathbf{A}^{2000} = \begin{bmatrix} 10^{-9} & \epsilon & 0.004 \\ 0 & 10^{-9} & \epsilon \\ 0 & 0 & 10^{-9} \end{bmatrix}$

More limits

- For any submultiplicative matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ and any $\mathbf{A} \in \mathbb{C}^{n,n}$ we have

$$\lim_{k\to\infty} \|\mathbf{A}^k\|^{1/k} = \rho(\mathbf{A}). \tag{14}$$